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Orthogonal Polynomials and Measures with Finitely Many Point Masses*

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Elementary proofs are given for a result of J. Geronimo and K. M. Case concerning the discrete spectrum of measures corresponding to orthogonal polynomials defined by a recerrence relation.

Let $p_n(x) = \gamma_n x^n + \cdots$, $\gamma_n > 0$, $n = 0, 1,\ldots$, be a sequence of orthonormalized polynomials with respect to some positive measure $d\alpha$ acting on the real line and having infinite support. Then the polynomials $p_n(x)$ satisfy the recurrence relation

$$xp_{n-1}(x) = \frac{\gamma_{n-1}}{\gamma_n} p_n(x) + \alpha_{n-1} p_{n-1}(x) + \frac{\gamma_{n-2}}{\gamma_{n-1}} p_{n-2}(x) \tag{1}$$

for $n = 1, 2,\ldots$, where $p_{-1}(x) \equiv 0$ and $p_0(x) \equiv \gamma_0$. The corresponding monic polynomials $P_n(x) = \gamma_n^{-1} p_n(x)$ can be defined by

$$xP_{n-1}(x) = P_n(x) + \alpha_{n-1} P_{n-1}(x) + \lambda_n P_{n-2}(x), \tag{2}$$

$n = 1, 2,\ldots$, $P_{-1}(x) \equiv 0$, $P_0(x) \equiv 1$ and $\lambda_n = \gamma_{n-2}^2/\gamma_{n-1}^2$.

By a result of J. Favard [4] there exists a one-to-one correspondence between polynomials defined either by (1) or by (2) and positive measures $d\alpha$ with $\int d\alpha = 1$, provided that the corresponding moment problem has a unique solution. It has been shown by O. Blumenthal [1] that if the polynomials $p_n(x)$ are orthogonal with respect to $d\alpha$ and satisfy (1) with

$$\lim_{n\to\infty} \alpha_n = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \frac{\gamma^{n-1}}{\gamma_n} = \frac{1}{2},$$

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then $[-1, 1] \subset \operatorname{supp}(d\alpha)$ and $\operatorname{supp}(d\alpha) \backslash [-1, 1]$ is a bounded and discrete set having at most two points of accumulation, namely, -1 and 1.

Because of several physical applications, recently much attention has been paid to orthogonal polynomials defined by recurrence relations. In particular, it has been proved in [7, Theorem 7.40] that if

$$\sum_{n=1}^{\infty} \left(|\alpha_n| + \left| \frac{\gamma_{n-1}}{\gamma_n} - \frac{1}{2} \right| \right) < \infty,$$

then $d\alpha$ can be written in the form

$$d\alpha(x) = w(x)\, dx + \sum_{i=1}^{\infty} \varepsilon_i \delta(x - t_i),$$

where w is a positive continuous function on $(-1, 1)$ and $t_i \notin (-1, 1)$. Since it is important to know whether $d\alpha$ has finitely or infinitely many point masses, the following recent result of J. Geronimo and K. M. Case [6] is of great significance.

THEOREM 1. *Let*

$$\sum_{n=1}^{\infty} n \left(|\alpha_n| + \left| \frac{\gamma_{n-1}}{\gamma_n} - \frac{1}{2} \right| \right) < \infty. \tag{3}$$

Then $\operatorname{supp}(d\alpha)$ contains at most finitely many points on the complement of $[-1, 1]$, and α is continuous at ± 1. Here

$$\alpha(x) = \int_{-\infty}^{x+0} d\alpha(t).$$

Geronimo and Case proved their result using a highly sophisticated and rather complicated argument. The purpose of the present paper is to give two elementary proofs of Theorem 1. Our first proof uses the chain sequence method. The second proof is based on some inequalities following from (1) when applied with $x = 1$. The feature of the second proof is that it does not require the introduction of other solutions of (1), which is the essence of both the Geronimo–Case and our first proof.

Our first proof is based on chain sequences and their relation to (2). We write

$$\mathcal{C}_n(x) = \lambda_{n+1}/(x - \alpha_{n-1})(x - \alpha_n).$$

Now let $\{P_n^{(k)}(x)\}_{n=0}^{\infty}$ denote the sequence of orthogonal polynomials defined by (2) after replacing α_{n-1} and λ_n by α_{n+k-1} and λ_{n+k}, respectively. A necessary and sufficient condition for the true interval of orthogonality of $\{P_n^{(k)}(x)\}_{n=0}^{\infty}$ to be a subset of $[-1, \infty)$ is $\alpha_n > -1$ for $n \geqslant k$ and $\{\mathcal{C}_n(-1)\}_{n=k+1}^{\infty}$ is a chain sequence [2, Theorem 1]. Under these conditions, supp($d\alpha$) contains at most k points smaller than -1 [2, Lemma 7].

Similarly, if $\alpha_n < 1$ for $n \geqslant m$ and if $\{\mathcal{C}_n(1)\}_{n=m+1}^{\infty}$ is a chain sequence, then supp($d\alpha$) contains at most m points larger than 1.

Conversely, if supp($d\alpha$) has only finitely many points outside $[-1, 1]$, then there is an N such that $|\alpha_n| < 1$ for $n \geqslant N$ and $\{\mathcal{C}_n(\pm 1)\}_{n=N+1}^{\infty}$ is a chain sequence [3, Theorem 1].

Thus, if $|\alpha_n| < 1$ for almost all n, then a necessary and sufficient condition for supp($d\alpha$) to contain finitely many points outside $[-1, 1]$ is that $\{\mathcal{C}_n(\pm 1)\}_{n=N}^{\infty}$ is a chain sequence for some N.

LEMMA 2. *Let* $\mathcal{C}_n = \frac{1}{4} + \varepsilon_n$ *where*

$$\sum_{n=1}^{\infty} n\,|\varepsilon_n| \leqslant \tfrac{1}{4}.$$

Then (i) $\{\mathcal{C}_n\}_{n=1}^{\infty}$ *is a chain sequence, and*

(ii) *if* $\{g_n\}_{n=1}^{\infty}$ *is any parameter sequence for* $\{\mathcal{C}_n\}$, *then*

$$g_n \leqslant \tfrac{1}{2} + 2 \sum_{v=1}^{\infty} |\varepsilon_{n+v}|, \qquad n \geqslant 0.$$

Proof. Let

$$\sigma_n = 4 \sum_{k=0}^{n} k\,|\varepsilon_k|.$$

Then $\sigma_0 = 0$, $0 \leqslant \sigma_n \leqslant 1$ for $n \geqslant 1$, so if we set

$$g_n = \frac{n + \sigma_n}{2(n+1)}$$

then $g_0 = 0$, $0 < g_n \leqslant \frac{1}{2}$ $(n \geqslant 1)$. Now consider the chain sequence

$$b_n = (1 - g_{n-1})\, g_n = \tfrac{1}{4} + \frac{(n+1)\sigma_n - \sigma_{n-1}(n + \sigma_n)}{4n(n+1)}.$$

Since $\sigma_n = \sigma_{n-1} + 4n\,|\varepsilon_n|$, we have

$$b_n = \tfrac{1}{4} + |\varepsilon_n| + \frac{\sigma_{n-1}(1 - \sigma_n)}{4n(n+1)}.$$

Thus $b_n \geqslant a_n$, so by Wall's "comparison test" [8, p. 86], $\{\mathcal{O}_n\}$ is a chain sequence. To prove (ii), let

$$h_n = \tfrac{1}{2} + 2 \sum_{v=1}^{\infty} |\varepsilon_{n+v}|, \qquad n \geqslant 0,$$

$$c_n(1-h_{n-1})h_n, \qquad n \geqslant 1.$$

Then $\{c_n\}_{n=1}^{\infty}$ is a chain sequence whose parameters satisfy $\frac{1}{2} \leqslant h_0 \leqslant 1$, $\frac{1}{2} \leqslant h_n < 1$ $(n \leqslant 1)$, and

$$\sum_{n=1}^{\infty} \frac{h_1 \cdots h_n}{(1-h_1)\cdots(1-h_n)} = \infty.$$

If $|\varepsilon_1| < \frac{1}{4}$, $c_n > 0$ for $n \geqslant 1$, so Wall's criterion [8, p. 82] shows that $\{h_n\}_{n=0}^{\infty}$ is the maximal parameter sequence. If $|\varepsilon_1| = \frac{1}{4}$, then $c_1 = 0$, so we apply Wall's criterion to $\{c_{n+1}\}_{n=1}^{\infty}$ and conclude $\{h_{n+1}\}_{n=0}^{\infty}$ is its maximal parameter sequence. Since $h_0 = 1$, it again follows that $\{h_n\}_{n=0}^{\infty}$ is the maximal parameter sequence for $\{c_n\}_{n=1}^{\infty}$. Finally, we observe that

$$c_n = \tfrac{1}{4} - |\varepsilon_n| - (h_{n-1} - \tfrac{1}{2})(h_n - \tfrac{1}{2}) \leqslant \tfrac{1}{4} - |\varepsilon_n| \leqslant a_n.$$

Thus by another result of Wall [8, Theorem 19.6], $g_n \leqslant h_n$.

First Proof of Theorem 1. For $x = \pm 1$, we have

$$a_n(x) - \frac{1}{4} = \frac{\lambda_{n+1} - (x-\alpha_{n-1})(x-\alpha_n)/4}{(x-\alpha_{n-1})(x-\alpha_n)}.$$

Thus there exists an $M > 0$ such that, for all n sufficiently large,

$$\begin{aligned} |a_n(x) - \tfrac{1}{4}| &\leqslant M\,|\lambda_{n+1} - \tfrac{1}{4}(x-\alpha_{n-1})(x-\alpha_n)| \\ &\leqslant M(|\lambda_{n+1} - \tfrac{1}{4}| + \tfrac{1}{4}|\alpha_n + \alpha_{n-1} \pm \alpha_n\alpha_{n-1}|). \end{aligned}$$

Therefore, $\sum n\,|a_n(x) - \frac{1}{4}| < \infty$, hence for sufficiently large N, $|a_n| < 1$ for $n \geqslant N$ and

$$\sum_{n=N}^{\infty} n\,|a_n(x) - \tfrac{1}{4}| \leqslant \tfrac{1}{4}.$$

By Lemma 2, $\{a_n(x)\}_{n=N+1}^{\infty}$ is a chain sequence; hence, there are at most finitely many points of supp$(d\alpha)$ outside $[-1,-1]$. It now follows [3, Theorem 1] that for $x = \pm 1$,

$$g_n(x) = 1 - \frac{P_{N+n+1}(x)}{(x-\alpha_{N+n})\,P_{N+n}(x)}, \qquad n \geqslant 0, \tag{4}$$

gives a sequence of parameters for $\{a_n(x)\}_{n=N+1}^{\infty}$. In terms of the corresponding orthonormal polynomials $p_n(x)$, (4) can be written

$$\frac{p_{N+n+1}^2(x)}{p_{N+n}^2(x)} = \lambda_{N+n+2}^{-1}[1 - g_n(x)]^2(x - \alpha_{N+n})^2.$$

Since $\lambda_{N+n+2} = a_{N+n+1}(x)(x - \alpha_{N+n-1})(x - \alpha_{N+n})$,

$$\frac{p_{N+n+1}^2(x)}{p_{N+n}^2(x)} = \frac{1 - g_n(x)}{g_{n+1}(x)} \frac{x - \alpha_{N+n}}{x - \alpha_{N+n-1}}.$$

According to Lemma 2, we have

$$\frac{1 - g_n(x)}{g_{n+1}(x)} \geqslant \frac{1 + f_n(x)}{1 + f_{n+1}(x)} = 1 + \frac{|4a_{n+N+1}(x) - 1|}{1 + f_{n+1}(x)},$$

where $f_m(x) = \sum_{v=N+1}^{\infty} |a_{m+v}(x) - \frac{1}{4}|$. Therefore

$$\frac{p_{n+n+1}^2(x)}{p_{N+n}^2(x)} \geqslant 1 + \frac{|4a_{n+N+1}(x) - 1|}{1 + f_{n+1}(x)} + \frac{\alpha_{N+n-1} - \alpha_{N+n}}{x - \alpha_{N+n-1}}$$
$$+ \frac{|4a_{n+N+1}(x) - 1| \, (\alpha_{N+n-1} - \alpha_{N+n})}{[1 + f_{n+1}(x)](x - \alpha_{N+n-1})}.$$

Because of (3), it now follows that

$$\liminf_{n\to\infty} n \left[\frac{p_{N+2+1}^2(x)}{p_{N+n}^2(x)} - 1\right] \geqslant 0.$$

Hence by Raabe's test $\sum p_n^2(x) = \infty$, and this implies α is continuous at x [5].

Remark. The least value of $N \geqslant 0$ such that $\alpha_n > -1$ for $n \geqslant N$ and $\{a_n(-1)\}_{n=N+1}^{\infty}$ is a chain sequence is an upper bound to the number of points smaller than -1 in supp($d\alpha$). A similar remark applies to $x = 1$.

Turning to our second proof of Theorem 1, we first establish a simple inequality derived from (1).

LEMMA 3. *Assume that the condition* (3) *is satisfied. Then there exists* n_0 *such that for* $N > m > n_0$ *the inequality*

$$|p_N(1)| \leqslant 3N \, |p_m(1) - p_{m-1}(1)|$$
$$+ |p_{m-1}(1)| \left\{1 + 2N \sum_{j=m-1}^{\infty} \right.$$
$$\left. \times \left[\left| \frac{\gamma_{j+1}}{\gamma_j}(1 - \alpha_j) - 2 \right| + \left| 1 - \frac{\gamma_{j+2}\gamma_j}{\gamma_{j+1}^2} \right| \right] \right\} \tag{5}$$

holds.

Proof. Let $n \geqslant m > 0$. Then from

$$p_{n+1}(1) - p_n(1) = p_m(1) - p_{m-1}(1) + \sum_{j=m}^{n} [p_{j+1}(1) - 2p_j(1) + p_{j-1}(1)]$$

we obtain

$$p_{k+1}(1) - p_{m-1}(1) = (k-m+2)[p_m(1) - p_{m-1}(1)] + \sum_{j=m}^{k} (k-j+1)[p_{j+1}(1) - 2p_j(1) + p_{j-1}(1)]$$

for $k \geqslant m > 0$. By the recurrence relation

$$p_{j+1}(1) = \frac{\gamma_{j+1}}{\gamma_j}(1-\alpha_j)\, p_j(1) - \frac{\gamma_{j+1}\gamma_{j-1}}{\gamma_j^2}\, p_{j-1}(1).$$

Hence

$$\begin{aligned} p_{k+1}(1) - p_{m-1}(1) &= (k-m+2)[p_m(1) - p_{m-1}(1)] \\ &\quad + (k-m+1)\left(1 - \frac{\gamma_{m+1}\gamma_{m-1}}{\gamma_m^2}\right) p_{m-1}(1) \\ &\quad + \sum_{j=m}^{k} \left\{(k-j+1)\left[\frac{\gamma_{j+1}}{\gamma_j}(1-\alpha_j) - 2\right]\right. \\ &\quad \left. + (k-j)\left(1 - \frac{\gamma_{j+2}\gamma_j}{\gamma_{j+1}^2}\right)\right\} p_j(1), \end{aligned}$$

that is,

$$\begin{aligned} p_{k+1}(1) - p_{m-1}(1) &= (k-m+2)[p_m(1) - p_{m-1}(1)] \\ &\quad + \sum_{j=m}^{k} \left\{(k-j+1)\left[\frac{\gamma_{j+1}}{\gamma_j}(1-\alpha_j) - 2\right]\right. \\ &\quad \left. + (k-j)\left(1 - \frac{\gamma_{j+2}\gamma_j}{\gamma_{j+1}^2}\right)\right\} [p_j(1) - p_{m-1}(1)] \\ &\quad + \left[(k-m+1)\left(1 - \frac{\gamma_{m+1}\gamma_{m-1}}{\gamma_m^2}\right)\right. \\ &\quad + \sum_{j=m}^{k} \left\{(k-j+1)\left[\frac{\gamma_{j+1}}{\gamma_j}(1-\alpha_j) - 2\right]\right. \\ &\quad \left.\left. + (k-j)\left(1 - \frac{\gamma_{j+2}\gamma_j}{\gamma_{j+1}^2}\right)\right\}\right] p_{m-1}(1). \end{aligned}$$

Consequently, for $k \geqslant m > 0$ the inequality

$$\frac{|p_{k+1}(1) - p_{m-1}(1)|}{k+1} \leqslant |p_m(1) - p_{m-1}(1)| + \sum_{j=m}^{k} j \left[\left| \frac{\gamma_{j+1}}{\gamma_j}(1-\alpha_j) - 2 \right| + \left| 1 - \frac{\gamma_{j+2}\gamma_j}{\gamma_{j+1}^2} \right| \right] \frac{|p_j(1) - p_{m-1}(1)|}{j} + |p_{m-1}(1)| \sum_{j=m-1}^{\infty} \left[\left| \frac{\gamma_{j+1}}{\gamma_j}(1-\alpha_j) - 2 \right| + \left| 1 - \frac{\gamma_{j+2}\gamma_j}{\gamma_{j+1}^2} \right| \right] \tag{6}$$

holds. By (3) there exists n_0 such that

$$\sum_{j=n_0}^{\infty} j \left[\left| \frac{\gamma_{j+1}}{\gamma_j}(1-\alpha_j) - 2 \right| + \left| 1 - \frac{\gamma_{j+2}\gamma_j}{\gamma_{j+1}^2} \right| \right] \leqslant \frac{1}{2}.$$

Let $N > m$ be given and suppose that $m > n_0$. Using (6) we otain

$$\max_{m<k\leqslant N} \frac{|p_k(1) - p_{m-1}(1)|}{k} \leqslant \frac{3}{2}|p_m(1) - p_{m-1}(1)| + \max_{m<j<N} \frac{|p_j(1) - p_{m-1}(1)|}{j} \times \sum_{j=m+1}^{\infty} j \left[\left| \frac{\gamma_{j+1}}{\gamma_j}(1-\alpha_j) - 2 \right| + \left| 1 - \frac{\gamma_{j+2}\gamma_j}{\gamma_{j+1}^2} \right| \right] + |p_{m-1}(1)| \sum_{j=m-1}^{\infty} \left[\left| \frac{\gamma_{j+1}}{\gamma_j}(1-\alpha_j) - 2 \right| + \left| 1 - \frac{\gamma_{j+2}\gamma_j}{\gamma_{j+1}^2} \right| \right]$$

$$\leqslant \frac{3}{2}|p_m(1) - p_{m-1}(1)| + \frac{1}{2} \max_{m<j\leqslant N} \frac{|p_j(1) - p_{m-1}(1)|}{j} + |p_{m-1}(1)| \sum_{j=m-1}^{\infty} \left[\left| \frac{\gamma_{j+1}}{\gamma_j}(1-\alpha_j) - 2 \right| + \left| 1 - \frac{\gamma_{j+2}\gamma_j}{\gamma_{j+1}^2} \right| \right].$$

Hence for $N > m > n_0$

$$\frac{|p_N(1) - p_{m-1}(1)|}{N} \leqslant 3|p_m(1) - p_{m-1}(1)| + 2|p_{m-1}(1)| \times \sum_{j=m-1}^{\infty} \left[\left| \frac{\gamma_{j+1}}{\gamma_j}(1-\alpha_j) - 2 \right| + \left| 1 - \frac{\gamma_{j+2}\gamma_j}{\gamma_{j+1}^2} \right| \right]$$

which implies (5).

THEOREM 4. *Let* (3) *be satisfied. Then there exist two real numbers A and B such that* $|A|+|B|>0$ *and*

$$\lim_{n\to\infty} \frac{p_n(1)}{nA+B} = 1.$$

Proof. We have

$$p_{n+1}(1) - p_n(1) = \sum_{k=0}^{n} [p_{k+1}(1) - 2p_k(1) + p_{k-1}(1)] + p_0(1).$$

Thus by the recurrence formula

$$p_{n+1}(1) - p_n(1) = \left[\frac{\gamma_{n+1}}{\gamma_n}(1-\alpha_n) - 2\right] p_n(1) + p_0(1)$$
$$+ \sum_{j=0}^{n-1} \left\{ \left[\frac{\gamma_{j+1}}{\gamma_j}(1-\alpha_j) - 2\right] + \left(1 - \frac{\gamma_{j+2}\gamma_j}{\gamma_{j+1}^2}\right) \right\} p_j(1). \quad (7)$$

By Lemma 3

$$\limsup_{k\to\infty} \frac{|p_k(1)|}{k} < \infty.$$

Hence by (7)

$$\lim_{n\to\infty} [p_{n+1}(1) - p_n(1)] = A < \infty$$

exists. If $A \neq 0$ then, since

$$\frac{p_n(1)}{n+1} = \frac{1}{n+1} \sum_{k=0}^{n} [p_k(1) - p_{k-1}(1)],$$

we obtain

$$\lim_{n\to\infty} p_n(1)/An = 1.$$

If $A = 0$, then

$$p_0(1) + \sum_{j=0}^{\infty} \left\{ \left[\frac{\gamma_{j+1}}{\gamma_j}(1-\alpha_j) - 2\right] + \left(1 - \frac{\gamma_{j+2}\gamma_j}{\gamma_{j+1}^2}\right) \right\} p_j(1) = 0$$

and by (7)

$$p_{n+1}(1) - p_n(1) = -\left(1 - \frac{\gamma_{n+2}\gamma_n}{\gamma_{n+1}^2}\right) p_n(1)$$
$$- \sum_{j=n+1}^{\infty} \left\{ \left[\frac{\gamma_{j+1}}{\gamma_j}(1-\alpha_j) - 2\right] + \left(1 - \frac{\gamma_{j+2}\gamma_j}{\gamma_{j+1}^2}\right) \right\} p_j(1).$$

Now let n_0 be defined by Lemma 3 and let $n > n_0$. Then we get

$$
\begin{aligned}
&|p_{n+1}(1) - p_n(1)| \\
&\quad \leqslant \left| 1 - \frac{\gamma_{n+2}\gamma_n}{\gamma_{n+1}^2} \right| |p_n(1)| + 3\,|p_{n+1}(1) - p_n(1)| \\
&\qquad \times \sum_{j=n+1}^{\infty} j \left\{ \left| \frac{\gamma_{j+1}}{\gamma_j}(1-\alpha_j) - 2 \right| + \left| 1 - \frac{\gamma_{j+2}\gamma_j}{\gamma_{j+1}^2} \right| \right\} \\
&\qquad + |p_n(1)| \sum_{j=n+1}^{\infty} \left\{ \left| \frac{\gamma_{j+1}}{\gamma_j}(1-\alpha_j) - 2 \right| + \left| 1 - \frac{\gamma_{j+2}\gamma_j}{\gamma_{j+1}^2} \right| \right\} \\
&\qquad + 2\,|p_n(1)| \sum_{k=n}^{\infty} \left\{ \left| \frac{\gamma_{k+1}}{\gamma_k}(1-\alpha_k) - 2 \right| + \left| 1 - \frac{\gamma_{k+2}\gamma_k}{\gamma_{k+1}^2} \right| \right\} \\
&\qquad \times \sum_{j=n+1}^{\infty} j \left\{ \left| \frac{\gamma_{j+1}}{\gamma_j}(1-\alpha_j) - 2 \right| + \left| 1 - \frac{\gamma_{j+2}\gamma_j}{\gamma_{j+1}^2} \right| \right\}.
\end{aligned}
$$

Let $n_1 > n_0$ be such that

$$
\sum_{j=n_1}^{\infty} j \left\{ \left| \frac{\gamma_{j+1}}{\gamma_j}(1-\alpha_j) - 2 \right| + \left| 1 - \frac{\gamma_{j+2}\gamma_j}{\gamma_{j+1}^2} \right| \right\} \leqslant \frac{1}{6}.
$$

For $n \geqslant n_1$ we obtain

$$
\begin{aligned}
&|p_{n+1}(1) - p_n(1)| \\
&\quad \leqslant \frac{14}{3} |p_n(1)| \sum_{k=n}^{\infty} \left\{ \left| \frac{\gamma_{k+1}}{\gamma_k}(1-\alpha_k) - 2 \right| + \left| 1 - \frac{\gamma_{k+2}\gamma_k}{\gamma_{k+1}^2} \right| \right\}.
\end{aligned}
$$

Since two consecutive orthogonal polynomials have no common zero, $p_n(1) \neq 0$ for $n \geqslant n_1$. Hence

$$
\begin{aligned}
&\sum_{n=n_1}^{\infty} \left| 1 - \frac{p_{n+1}(1)}{p_n(1)} \right| \\
&\quad \leqslant \frac{14}{3} \sum_{n=n_1}^{\infty} \sum_{k=n}^{\infty} \left\{ \left| \frac{\gamma_{k+1}}{\gamma_k}(1-\alpha_k) - 2 \right| + \left| 1 - \frac{\gamma_{k+2}\gamma_k}{\gamma_{k+1}^2} \right| \right\} \\
&\quad \leqslant \frac{14}{3} \sum_{n=1}^{\infty} n \left\{ \left| \frac{\gamma_{n+1}}{\gamma_n}(1-\alpha_n) - 2 \right| + \left| 1 - \frac{\gamma_{n+2}\gamma_n}{\gamma_{n+1}^2} \right| \right\} < \infty.
\end{aligned}
$$

Therefore

$$
\prod_{n=n_1}^{\infty} \frac{p_{n+1}(1)}{p_n(1)}
$$

converges, which is equivalent to the existence of

$$\lim_{n\to\infty} p_n(1) = B,$$

where $0 \neq B < \infty$.

Second Proof of Theorem 1. By Theorem 4

$$\sum_{k=0}^{\infty} p_k^2(1) = \infty.$$

Therefore by a classical result from the theory of moment problems [5] α must be continuous at 1. Now we will show that $\text{supp}(d\alpha) \cap (1, \infty)$ is finite which means that α has no more than a finite number of jumps in $(1, \infty)$. It follows from Theorem 4 that there exists N such that for every $n > N$

$$\text{sign } p_n(1) = \text{sign } p_N(1) \neq 0.$$

Let Z_n denote the number of zeros x_{kn} of $p_n(x)$ in $(1, \infty)$. Since the zeros of $p_k(x)$ separate the zeros of $p_{n+1}(x)$ and the leading coefficient of $p_n(x)$ is positive for every n we obtain

$$Z_n = Z_N \leqslant N$$

for $n \geqslant N$. Furthermore, for every $x \in \text{supp}(d\alpha)$ one can choose a sequence $\{k_n\}$ such that $1 \leqslant k_n \leqslant n$ and

$$\lim_{n\to\infty} x_{k_n,n} = x.$$

Hence $\text{supp}(d\alpha) \cap (1, \infty)$ contains $Z_N \leqslant N$ points. Finally, applying what has just been proved to $\alpha^*(x) = -\alpha(-x)$ we get that α is continuous at -1 and it has at most a finite number of jumps in $(-\infty, -1)$.

In Lemma 2 it was shown that a sufficient condition for $\{a_n\}$ to be a chain sequence is

$$\sum_{n=0}^{\infty} n\,|a_n - \tfrac{1}{4}| \leqslant \tfrac{1}{4}.$$

On the other hand, if $a_n \geqslant \frac{1}{4}$, a necessary condition is

$$\sum_{k=1}^{n} (a_k - \tfrac{1}{4}) \leqslant \tfrac{3}{8}$$

[4]. The constant $\frac{3}{8}$ can be improved.

THEOREM 5. *If $\{a_n\}$ is a chain sequence such that $a_n \geqslant \frac{1}{4}$ then*

$$\sum_{k=1}^{n} (a_k - \tfrac{1}{4}) \leqslant \tfrac{1}{4}.$$

Proof. Let $a_k = (1 - m_{k-1}) m_k$ with $m_0 = 0$ and $0 < m_k < 1$. Since $a_k \geqslant \frac{1}{4}$, $0 < m_k \leqslant \frac{1}{2}$ for $k = 1, 2, \ldots$ [8]. We have

$$\begin{aligned}\sum_{k=1}^{n} a_k &= \sum_{k=1}^{n} m_k(1 - m_{k-1}) \\ &= \sum_{k=1}^{n} [\tfrac{1}{2} + (m_k - \tfrac{1}{2})][\tfrac{1}{2} - (m_{k-1} - \tfrac{1}{2})] \\ &= \sum_{k=1}^{n} [\tfrac{1}{4} + \tfrac{1}{2}(m_k - m_{k-1}) - (m_k - \tfrac{1}{2})(m_{k-1} - \tfrac{1}{2})].\end{aligned}$$

Therefore

$$\sum_{k=1}^{n} (a_k - \tfrac{1}{4}) = m_n/2 - \sum_{k=1}^{n} (m_k - \tfrac{1}{2})(m_{k-1} - \tfrac{1}{2}) \leqslant m_n/2 \leqslant \tfrac{1}{4}.$$

It can easily be seen from the proof that

$$\sum_{k=1}^{n} [a_k - \tfrac{1}{4}] = \tfrac{1}{4}$$

for $n = 1, 2, \ldots$ if and only if $a_1 = \frac{1}{2}$ and $a_k = \frac{1}{4}$ for $k = 2, 3, \ldots$. Hence the constant $\frac{1}{4}$ cannot be improved. The sequence $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \ldots$ is not a chain sequence so the condition in Theorem 5 is not sufficient for $\{a_n\}$ to be a chain sequence.

REFERENCES

1. O. BLUMENTHAL, "Über die Entwicklung einer willkürlichen Funktion nach den Nennern des Kettenbruches für $\int_{-\infty}^{\infty} [\phi(\xi)/(z - \xi)]\, d\xi$," Inaugural dissertation, Göttingen, 1898.
2. T. S. CHIHARA, Chain sequences and orthogonal polynomials, *Trans. Amer. Math. Soc.* **104** (1962), 1 – 16.
3. T. S. CHIHARA, Orthogonal polynomials whose zeros are dense in intervals, *J. Math. Anal. Appl.* **24** (1978), 362–371.
4. J. FAVARD, Sur les polynomes de Tchebicheff, *C. R. Acad. Sci. Paris* **200** (1935), 2052–2053.
5. G. FREUD, "Orthogonal Polynomials," Pergamon, New York, 1971.
6. J. S. GERONIMO AND K. M. CASE, Scattering theory and polynomials orthogonal on the real line, *Trans. Amer. Math. Soc.* **258** (1980), 467–494.
7. P. G. NEVAI, "Orthogonal Polynomials," *Mem. Amer. Math. Soc.* **213** (1979).
8. H. S. WALL, "Analytic Theory of Continued Fractions," Van Nostrand, Princeton, N. J., 1948.